THE SPECTRA OF NON-ELLIPTIC OPERATORS

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ABSTRACT

We discuss the spectrum of the minimal operator corresponding to a constant coefficient partial differential operator on $L^p(E^n)$. We then study effects on the spectrum by various perturbations.

1. **Introduction.** Let $P(D)$ be a constant coefficient partial differential operator in E^n. Acting on the set C_0^∞ of smooth functions with compact supports, $P(D)$ is closable in $L^p(E^n)$, $1 \leqq p \leqq \infty$. In this paper we are concerned with certain spectral properties of its closure P_0 in $L^p(E^n)$.

For $p = 2$, $\sigma(P_0)$ consists of the closure of the set of values taken on by $P(\xi)$ for ξ real. We were able to obtain the same result for $1 < p < \infty$ under the assumptions

$$P^{(\mu)}(\xi)/P(\xi) = 0(|\xi|^{-a|\mu|}) \text{ as } |\xi| \to \infty \tag{1.1}$$

$$1/P(\xi) = 0(|\xi|^{-b}) \quad \text{as } |\xi| \to \infty, \tag{1.2}$$

ξ real, for $b > (1-a)l$, $2l > n$, where $P^{(\mu)}(\xi)$ denotes a derivative of $P(\xi)$ of order $|\mu|$.

We then obtain sufficient conditions on an operator $Q(D)$ to imply that $D(P_0) \subseteq D(Q_0)$ and on a function $q(x)$ that $D(P_0) \subseteq D(qQ_0)$ and that qQ_0 be P_0-compact. This enables us to study the essential spectrum of an operator of the form

$$L(x, D) = \sum_1^r a_j(x) Q_j(D).$$

Most previous work on these questions concerned elliptic operators. A partial list of contributors includes Balslev [1], Browder [2], Birman [3], Glazman [4], Kato [5], Rejto [6], Stummel [8], Wolf [9], and Schechter [10]. See also the book [11] by Glazman and the authors quoted there.

Results for non-elliptic operators were obtained by Niznik [19] and Martirosjan [7].

2. **The Main Results.** Let $P(\xi)$ be a polynomial of degree m in the variables $\xi = (\xi_1, \cdots, \xi_n)$. If we replace ξ by $D = (D_1, \cdots, D_n)$, where

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$$D_j = -i\partial/\partial x_j, \qquad 1 \leqq j \leqq n,$$

we obtain a constant coefficient partial differential operator $P(D)$. Let p satisfy $1 \leqq p \leqq \infty$. The *minimal* operator P_0 in $L^p = L^p(E^n)$ corresponding to $P(D)$ is defined as follows. A function $u \in L^p$ is in $D(P_0)$ and $P_0 u = f$ if there is a sequence $\{u_k\}$ of functions in C_0^∞(the set of infinitely differentiable functions with compact supports) such that $u_k \to u$ and $P(D)u_k \to f$ in L^p. Concerning the spectrum $\sigma(P_0)$ of P_0 we have

THEOREM 2.1. *In order that λ be in $\rho(P_0)$ it is necessary, and for $p = 2$ also sufficient, that $P(\xi) - \lambda$ be bounded away from zero for ξ real.*

REMARK 2.2. In general the non-vanishing of $P(\xi)-\lambda$ for real ξ does not imply that it is bounded away from zero. For instance, if

$$P(\xi) = (\xi_1\xi_2 - 1)^2 + \xi_2^2 + \cdots + \xi_n^2,$$

then $P(\xi) \neq 0$ for all real vectors ξ. But if we take $\xi_2 = 1/\xi_1$ and $\xi_j = 0$ for $j > 2$, then $P(\xi) \to 0$ as $|\xi_1| \to \infty$.

REMARK 2.3. By Theorem 2.1 we have for $p = 2$ that $\sigma(P_0)$ consists of those λ such that there is a sequence $\{\xi^{(k)}\}$ of real vectors satisfying $P(\xi^{(k)}) \to \lambda$ as $k \to \infty$.

In order to describe our results for $p \neq 2$, we let $\mu = (\mu_1, \cdots, \mu_n)$ be a multi-index of non-negative integers. Set $|\mu| = \mu_1 + \cdots + \mu_n$ and

$$P^{(\mu)}(\xi) = \partial^{|\mu|}P(\xi)/\partial\xi_1^{\mu_1} \cdots \partial\xi_n^{\mu_n}.$$

We assume that

$$P^{(\mu)}(\xi)/P(\xi) = O(|\xi|^{-a|\mu|}) \quad \text{as} \quad |\xi| \to \infty, \quad \text{each } \mu, \tag{2.1}$$

and

$$1/P(\xi) = O(|\xi|^{-b}) \qquad \text{as} \quad |\xi| \to \infty \tag{2.2}$$

for real vectors ξ, where $a \geqq 0$ and $b > 0$. We have

THEOREM 2.4. *Let l be an integer $> n/2$ and assume that $P(\xi)$ satisfies* (2.1) *and* (2.2) *for $b \geqq (1-a)l$. If $1 < p < \infty$, then $\lambda \in \rho(P_0)$ if and only if $P(\xi) \neq \lambda$ for all real vectors ξ.*

REMARK 2.5. Many types of operators satisfy (2.1) and (2.2). If $a > 0$, $P(\xi)$ is hypo-elliptic. If $a = 1$, $P(\xi)$ is elliptic. We always have $a \leqq 1$ and $b \geqq ma$.

REMARK 2.6. One can define a *maximal* operator P_p corresponding to $P(D)$ in L^p. We say that a function $u \in L^p$ is in $D(P_p)$ and $P_p u = f$ if

$$(u, \bar{P}(D)\phi) = (f, \phi), \qquad \phi \in C_0^\infty, \tag{2.3}$$

where $\bar{P}(\xi)$ is the polynomial the coefficients of which are the complex conjugates of those of $P(\xi)$. It has been proved by Goldstein [12] that $P_p = P_0$ for $1 \leqq p < \infty$. Hence all of the statements made so far apply to P_p as well.

Next let $q(x)$ be a function defined on E^n, and let V be the set of those functions $u \in L^p$ such that $qu \in L^p$. We can consider multiplication by q as an operator on L^p with domain V. This operator is closed; denote it also by q. We shall give sufficient conditions for $D(P_0) \subseteq D(q)$ and for q to be P_0-compact. For arbitrary operators A, B we say that B is A-compact if $D(A) \subseteq D(B)$ and

$$\| x_k \| + \| Ax_k \| \leqq C, \qquad x_k \in D(A), \tag{2.4}$$

implies that $\{Bx_k\}$ has a convergent subsequence.

THEOREM 2.7. *Suppose $P(\xi)$ satisfies (2.1) and (2.2) for real ξ, with $b > (1-a)n + a$. Let k_0 denote the smallest integer $\geqq 0$ such that $a\, k_0 > n - b$. Assume that $1 \leqq p < \infty$ and that $q(x)$ is a function locally in L^p such that $M_{\alpha\, p}(q) < \infty$ for some α satisfying*

$$-n < \alpha < p(n - k_0) - n, \tag{2.5}$$

where

$$M_{\alpha,p}(q) = \sup_y \int_{|x-y|<1} | q(x) |^p | x - y |^\alpha dx. \tag{2.6}$$

Assume also that $\rho(P_0)$ is not empty. Then $D(P_0) \subseteq D(q)$.

THEOREM 2.8. *If the hypotheses of Theorem 2.7 are satisfied and*

$$\int_{|x-y|<1} | q(x) |^p dx \to 0 \quad \text{as} \quad | y | \to \infty, \tag{2.7}$$

then q is P_0-compact.

REMARK 2.9. For an arbitrary operator A on a Banach space, there are at least seven definitions for the essential spectrum $\sigma_e(A)$ of A (cf. [13, 14]). Most of them coincide for a self-adjoint operator in Hilbert space. For these one has

$$\sigma_e(A + B) = \sigma_e(A) \tag{2.8}$$

whenever B is A-compact. Thus under the hypotheses of Theorem 2.8 we have

$$\sigma_e(P_0 + q) = \sigma_e(P_0). \tag{2.9}$$

Moreover, under every definition of essential spectrum one has

$$\sigma_e(P_0) = \sigma(P_0). \tag{2.10}$$

Hence

$$\sigma(P_0 + q) \supseteq \sigma_e(P_0 + q) = \sigma(P_0). \tag{2.11}$$

Let $P(\xi)$ and $Q(\xi)$ be polynomials and let P_0 and Q_0 be the minimal operators corresponding to $P(D)$ and $Q(D)$, respectively. We give conditions under which one has $D(P_0) \subseteq D(Q_0)$.

THEOREM 2.10. *A necessary, and for $p = 2$ also sufficient, condition that $D(P_0) \subseteq D(Q_0)$ is that*

$$|Q(\xi)| \leqq C(|P(\xi)| + 1), \qquad \xi \text{ real.} \tag{2.12}$$

When $p \neq 2$ we have a weaker result.

THEOREM 2.11. *Suppose that $P(\xi)$ satisfies* (2.1) *and that*

$$Q(\xi)/P(\xi) = 0(|\xi|^{-c}) \text{ as } |\xi| \to \infty, \quad \xi \text{ real.} \tag{2.13}$$

Assume that $c \geqq (1 - a)l$, where l is an integer $> n/2$. If $1 < p < \infty$ and $\rho(P_0)$ is not empty, then $D(P_0) \subseteq D(Q_0)$.

The next two theorems are concerned with the operator $q\, Q_0$.

THEOREM 2.12. *Suppose $1 \leqq p < \infty$ and that* (2.1) *and* (2.13) *hold with $c > (1 - a)n + a$. Assume that q is locally in L^p and that $M_{\alpha,p}(q) < \infty$ for some α satisfying*

$$-n < \alpha < p(n - k_0) - n, \tag{2.14}$$

where k_0 is the smallest integer $\geqq 0$ satisfying $k_0\, a > n - c$. If $\rho(P_0)$ is not empty, then $D(P_0) \subseteq D(q\, Q_0)$.

THEOREM 2.13. *If* (2.7) *holds in addition to the hypotheses of Theorem* 2.12, *then the operator $q\, Q_0$ is P_0-compact.*

Every variable coefficient partial differential operator is of the form

$$L(x, D) = \sum_{j=1}^{r} a_j(x) Q_j(D), \tag{2.15}$$

where the $Q_j(D)$ are constant coefficient operators and the $a_j(x)$ are functions of coordinates. We can define the minimal operator L_0 corresponding to $L(x, D)$ in the same way as was done for constant coefficient operators.

THEOREM 2.14. *Assume that there are constants a_j such that the constant coefficient operator $P(D) = \Sigma a_j\, Q_j(D)$ satisfies* (2.1) *and*

$$Q_j(\xi)/P(\xi) = 0(|\xi|^{-c_j}) \text{ as } |\xi| \to \infty \tag{2.16}$$

for each j, where $c_j > (1 - a)n + a$. Suppose $1 \leqq p < \infty$ and

(2.17) $$M_{\alpha j,p}[a_j(x)] < \infty, \qquad 1 \leqq j \leqq r,$$

where

(2.18) $$-n < \alpha_j < p(n - k_j) - n$$

and k_j is the smallest integer $\geqq 0$ satisfying $ak_j < n - c_j$. If $\rho(P_0)$ is not empty and

(2.19) $$\int_{|x-y|<1} |a_j(x) - a_j|^p dx \to 0 \text{ as } |y| \to \infty, \qquad 1 \leqq j \leqq r,$$

then $L_0 - P_0$ is P_0-compact. Thus

$$\sigma(L_0) \supseteq \sigma_e(L_0) = \sigma_e(P_0) = \sigma(P_0)$$

for those definitions of essential spectrum discussed in Remark 2.9.

3. Proofs.

Proof of Theorem 2.1. Without loss of generality, we may assume that $\lambda = 0$. If $P(\xi)$ is not bounded away from zero, there is a sequence $\{\xi^{(k)}\}$ of real vectors such that $P(\xi^{(k)}) \to 0$ as $k \to \infty$. Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that

(3.1) $$\varepsilon_k^{|\mu|} P^{(\mu)}(\xi^{(k)}) \to 0 \text{ as } k \to \infty$$

holds for each μ, and let ψ be a function in C_0^∞ such that $\|\psi\| = 1$ (the norm is that of L^p). Set

(3.2) $$\phi_k(x) = \varepsilon_k^{n/p} e^{i\xi^{(k)}x} \psi(\varepsilon_k x), \qquad k = 1, 2, \cdots,$$

where $1/\infty$ is to be interpreted as 0. Thus

(3.3) $$\|\phi_k\| = 1, \qquad k = 1, 2, \cdots.$$

Now by Leibnitz's formula (cf. [15])

$$P(D)\phi_k = \varepsilon_k^{n/p} e^{i\xi^{(k)}x} \sum_\mu \varepsilon_k^{|\mu|} P^{(\mu)}(\xi^{(k)}) \psi_\mu(\varepsilon_k x)/\mu!,$$

where $\psi_\mu(x) = D^\mu \psi(x) = (-1)^{|\mu|} \partial^{|\mu|} \psi(x) / \partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}$ and $\mu! = \mu_1! \cdots \mu_n!$. Since

$$\| \varepsilon_k^{n/p} \psi_\mu(\varepsilon_k x) \| = \| \psi_\mu \|,$$

we have by (3.1)

$$P(D)\phi_k \to 0 \text{ as } k \to \infty$$

in L^p. This shows that $0 \in \sigma(P_0)$.

Next suppose that there is a constant $c_0 > 0$ such that

(3.4) $$|P(\xi)| \geqq c_0, \qquad \xi \text{ real}.$$

Let S be the set of infinitely differentiable functions v on E^n such that

$$|x|^j |D^\mu v(x)|$$

is bounded for each j and μ. If $f \in S$, then its Fourier transform Ff is also in S. By (3.4) the same is true of Ff/P. Thus there is a $u \in S$ satisfying

(3.5) $$Fu = Ff/P.$$

(3.6) $$P(D)u = f$$

Thus

In particular

(3.7) $$(u, \bar{P}(D)\phi) = (f, \phi), \qquad \phi \in C_0^\infty,$$

showing that $u \in D(P_0)$. By Goldstein's result [12], we have $u \in D(P_0)$. Moreover, by (3.4) and (3.5)

$$|Fu| \leqq |Ff|/c_0,$$

which implies

$$\| Fu \| \leqq \| Ff \|/c_0.$$

If $p = 2$, Parseval's identity then gives

(3.8) $$\| u \| \leqq \|f\|/c_0.$$

Since S is dense in L^p and P_0 is a closed operator, this shows that for each $f \in L^p$ there is a unique $u \in D(P_0)$ such that $P_0 u = f$ and (3.8) holds. Thus $0 \in \rho(P_0)$ and the proof of Theorem 2.1 is complete.

Before proving Theorem 1.4, let me give the

Proof of Theorem 2.11. Assume $0 \in \rho(P_0)$. I am going to prove

(3.9) $$\| Q(D)v \| \leqq C\|P(D)v\|, \qquad v \in S.$$

Since $C_0^\infty \subset S$, it follows from (3.9) that $D(P_0) \subseteq D(Q_0)$. To prove (3.9) note that

$$F[Q(D)v] = \frac{Q(\xi)}{P(\xi)} F[P(D)v].$$

Inequality (3.9) will follow if we can show that $Q(\xi)/P(\xi)$ is a multiplier in L^p for for $1 < p < \infty$. Now I claim that

(3.10) $$Q^{(\mu)}(\xi)/P(\xi) = O(|\xi|^{-a|\mu|-c}) \text{ as } |\xi| \to \infty.$$

Assume this for the moment. An easy induction shows that $D^{\mu}(Q/P)$ is a sum of terms of the form

$$\text{constant } Q^{(\mu^{(1)})}(\xi)P^{(\nu^{(1)})}(\xi)\cdots P^{(\nu^{(t)})}(\xi)/P(\xi)^{1+t},$$

where $\mu^{(1)}+\nu^{(1)}+\cdots\nu^{(t)}=\mu$. Thus

$$(3.11)\qquad |D^{\mu}(Q/P)|\leqq C|\xi|^{-a|\mu|-c}.$$

Since $c\geqq(1-a)l$, we have $c+a|\mu|\geqq|\mu|\leqq l$. By a generalization of Mikhlin's theorem, this shows that Q/P is a multiplier in L^p (cf. [16, 17, 18]).

It remains to prove (3.10). For each μ there are vectors $\theta^{(1)},\cdots,\theta^{(r)}$ and coefficients $\gamma_1,\cdots,\gamma_r$ such that $|\theta^{(j)}|=1$ and

$$(3.12)\qquad t^{|\mu|}Q^{(\mu)}(\xi)=\Sigma\gamma_j Q(\xi+t\theta^{(j)})$$

holds for all real ξ and $t\geqq 1$ (cf. [15]). Set $t=|\xi|^a/2$. Then for $|\xi|\geqq 1$ we have

$$|\xi+t\theta^{(j)}|\geqq|\xi|\frac{a}{}-\frac{1}{2}|\xi|\ \geqq|\xi|/2.$$

Now by (2.1), (2.13) and (3.12)

$$\begin{aligned}|\xi|^{a|\mu|}|Q^{(\mu)}(\xi)|&\leqq C\,\Sigma|P(\xi+t\theta^{(j)})|\quad|\xi+t\theta^{(j)}|^{-c}\\&\leqq C\sum_{\nu}|P^{(\nu)}(\xi)||\xi|^{a|\nu|-c}\\&\leqq C|P(\xi)||\xi|^{-c}\end{aligned}$$

for $|\xi|$ large. This gives (3.10) and the proof of Theorem 2.11 is complete.

We can now give the

Proof of Theorem 2.4. By Theorem 2.1 it suffices to show that if $P(\xi)\neq\lambda$ for each ξ, then $\lambda\in\rho(P_0)$. We may take $\lambda=0$. By (2.2) $|P(\xi)|\to\infty$ as $|\xi|\to\infty$. Thus if $P(\xi)\neq 0$ for real ξ, there is a constant $c_0>0$ such that (3.4) holds. By the reasoning in the proof of Theorem 2.1 we see that $R(P_0)$ is dense in L^p. Moreover, if we take $Q(\xi)=1$, the hypotheses of Theorem 2.11 are satisfied with $c=b$. Hence by (3.9)

$$(3.12)\qquad \|v\|\leqq C\|P(D)v\|,\qquad v\in S,$$

which shows that $R(P_0)$ is closed in L^p and that $N(P_0)=\{0\}$. Hence $0\in\rho(P_0)$, and the proof is complete.

Note that Theorems 2.7 and 2.8 are special cases of Theorems 2.12 and 2.13, respectively.

Proof of Theorem 2.10. Let ξ be a real vector in E^n and let ψ be a function in S such that $\| \psi \| = 1$. For $\varepsilon > 0$ set

$$\phi_\varepsilon(x) = \varepsilon^{n/p} e^{i\xi x} \psi(\varepsilon x).$$

Then

$$P(D)\phi_\varepsilon(x) = \varepsilon^{n/p} e^{i\xi x} \sum_\mu \varepsilon^{|\mu|} P^{(\mu)}(\xi) \psi_\mu(\varepsilon x)/\mu!,$$

where $\psi_\mu = D^\mu \psi$. Thus

$$\| P(D)\phi_\varepsilon \| \to | P(\xi) | \text{ as } \varepsilon \to 0. \tag{3.13}$$

Similarly,

$$\| Q(D)\phi_\varepsilon \| \to | Q(\xi) | \text{ as } \varepsilon \to 0. \tag{3.14}$$

Now if $D(P_0) \subseteq D(Q_0)$, we see from the fact that they are both closed operators that

$$\| Q_0 v \| \leqq C(\| P_0 v \| + \| v \|), \qquad v \in D(P_0).$$

Hence

$$\| Q(D)\phi_\varepsilon) \| \leqq C(\| P(D)\phi_\varepsilon \| + \| \phi_\varepsilon \|). \tag{3.15}$$

Since $\| \phi_\varepsilon \| = 1$, we obtain (2.12) by letting $\varepsilon \to 0$ in (3.15) and employing (3.13) and (3.14).

Conversely, assume that (2.12) holds and that $p = 2$. Then

$$| Q(\xi)Fv |^2 \leqq C(| P(\xi)Fv |^2 + | Fv |^2), \qquad v \in S.$$

Integrating with respect to ξ, we have by Parseval's identity

$$\| Q(D)v \| \leqq C(\| P(D)v \| + \| v \|), \qquad v \in S. \tag{3.16}$$

Now let v be any function in $D(P_0)$. Then there is a sequence $\{v_k\}$ of functions in S such that $v_k \to v$ and $P(D)v_k \to P_0 v$ in L^2. By (3.16), $Q(D)v_k$ converges in L^2 to some function w. Thus $v \in D(Q_0)$ and $Q_0 v = w$.

In proving Theorems 2.12 and 2.13 we shall make use of the following results.

THEOREM 3.1. *Let k_0 be an integer satisfying $0 \leqq k_0 < n$, and let w be a function in $C^{n+1}(E^n)$ satisfying*

$$\| D^\mu w \|_1 \leqq K_1, \qquad | \mu | = k_0$$

$$\| D^\mu w \|_1 \leqq K_2, \qquad | \mu | = n + 1.$$

Suppose $1 \leqq p < \infty$ and let α be a number satisfying

$$- n < \alpha < p(n - k_0) - n. \tag{3.17}$$

Let $q(x)$ be a function locally in L^p_s and let T be the operator defined by

$$Tf = q[F^{-1}(w) * f]. \tag{3.18}$$

Then

$$\| Tf \| \leqq C(K_1 + K_2)[M_{\alpha,p}(q)]^{1/p} \|f\|, \qquad f \in L^p_s \tag{3.19}$$

where the constant C depends only on n, k_0, α and p.

Proof. Set

$$G(x) = F^{-1}(w).$$

Then by integration by parts

$$|x^\mu G(x)| = |F^{-1}(D^\mu w)| \leqq \| D^\mu w \|_1.$$

Thus

$$|G(x)| \leqq K_1 |x|^{-k_0}, \qquad x \in E^n \tag{3.20}$$

and

$$|G(x)| \leqq K_2 |x|^{-n-1}, \qquad x \in E^n. \tag{3.21}$$

Assume first that $1 < p < \infty$. Then for $v \in S$

$$(Tf, v) = \int\int q(x)G(x-y)f(y)\overline{v(x)}\,dxdy.$$

Thus

$$\begin{aligned} |(Tf,v)| &\leqq \int\int_{|x-y|<1} + \int\int_{|x-y|>1} \\ &\leqq \left(\int\int_{|x-y|<1} |q(x)|^p |G(x-y)|^{\beta p} |f(y)|^p\,dxdy \right)^{1/p} \\ &\quad \cdot \left(\int\int_{|x-y|<1} |G(x-y)|^{(1-\beta)p'} |v(x)|^{p'}\,dxdy \right)^{1/p'} \\ &\quad + \left(\int\int_{|x-y|>1} |q(x)|^p |G(x-y)| |f(y)|^p\,dxdy \right)^{1/p} \\ &\quad \cdot \left(\int\int_{|x-y|>1} |G(x-y)| |v(x)|^{p'}\,dxdy \right)^{1/p'} \end{aligned} \tag{3.22}$$

for any β satisfying $0 \leqq \beta \leqq 1$. Now in general

$$M_{\gamma,p}(q) \leqq M_{\delta,p}(q), \qquad \gamma \geqq \delta \tag{3.23}$$

and

$$M_{0,p}(q) \leqq CM_{\gamma,p}(q), \tag{3.24}$$

where C depends only on γ and n. Thus we may assume without loss of generality that

$$-k_0 p \leqq \alpha \leqq 0.$$

We take

$$\beta = |\alpha|/k_0 p.$$

Then $0 \leqq \beta \leqq 1$ and by (3.17)

$$1 - (n/p'k_0) < \beta < n/p\, k_0.$$

Thus $(1 - \beta)p'k_0 < n$, so that

$$\int_{|z|<1} |G(z)|^{(1-\beta)p'} dx \leqq K_1 \int_{|z|<1} |z|^{-(1-\beta)p'k_0}\, dz. \tag{3.25}$$

Moreover by our choice of β

$$\int_{|x-y|<1} |q(x)|^p |G(x-y)|^{\beta p} dx \leqq K_1 M_{\alpha,p}(q). \tag{3.26}$$

By (3.21)

$$\int_{|z|>1} |G(z)|\, dz \leqq K_2 \int_{|z|>1} |z|^{-n-1}\, dz. \tag{3.27}$$

I claim further that

$$\int_{|x-y|>1} |q(x)|^p |G(x-y)|\, dx \leqq C\, K_2\, M_{0,p}(q), \tag{3.28}$$

where C depends only on n. Assuming this for the moment, we have by (3.22), (3.24)–(3.28),

$$|(Tf, v)| \leqq C(K_1 + K_2)[M_{\alpha,p}(q)]^{1/p} \|f\|_p \|v\|_{p'},$$

which implies (3.19). The case $p = 1$ is easily disposed of. Inequality (3.17) becomes $-n < \alpha < -k_0$. Thus by (3.20) and (3.28)

$$\|Tf\|_1 \leqq \iint_{|x-y|<1} |q(x)||G(x-y)||f(y)|\,dxdy$$
$$+ \iint_{|x-y|>1} |g(z)||G(x-y)||f(y)|\,dxdy$$
$$\leqq K_1 M_{\alpha,p}(q)\|f\| + CK_2 M_{0,p}(q)\|f\|,$$

which implies (3.19) in this case as well.

It therefore remains only to prove (3.18). Now

$$\int_{|x-y|>1} |q(x)|^p |G(x-y)|\,dx = \sum_{k=1}^{\infty} \int_{k<|x-y|<k+1}$$
$$\leqq K_2 \sum_{k=1}^{\infty} k^{-n-1} \int_{k<|x-y|<k+1} |q(x)|^p dx.$$

But there is a constant C depending only on n such that

$$\int_{k<|x-y|<k+1} |q(x)|^p dx \leqq Ck^{n-1} M_{0,p}(q). \tag{3.29}$$

Thus

$$\int_{|x-y|>1} |q(x)|^p |G(x-y)|\,dx \leqq CK_2 M_{0,p}(q) \sum_{k=1}^{\infty} k^{-2}, \tag{3.30}$$

which is merely (3.28). This completes the proof.

LEMMA 3.2. *Let ϕ be a function in C_0^∞ and let Ω be a bounded subset of E^n. Then the operator*

$$Af = F^{-1}(\phi) * f$$

is a compact operator from L^p to $C(\bar{\Omega})$.

Proof. Since Af is a smoth function, we have

$$D_j Af = D_j F^{-1}(\phi) * f = -F^{-1}(\xi_j\phi) * f.$$

Since $F^{-1}(\phi)$ and $F^{-1}(\xi_j\phi)$ are in L^r for any r, we have by Young's inequality

$$\|Af\|_\infty + \Sigma\|D_j Af\|_\infty \leqq C\|f\|_p. \tag{3.31}$$

Now let $\{f_k\}$ be a sequence of functions in L^p satisfying

$$\|f_k\|_p \leqq C.$$

By (3.31) $\{Af_k\}$ is a uniformly bounded, equi-continuous sequence of functions on $\bar{\Omega}$. Thus it has a convergent subsequence.

Proof of Theorem 2.12 We may assume that $0 \in \rho(P_0)$. Since the hypotheses of Theorem 2.11 are fulfilled, inequality (3.9) holds. Moreover, I claim that

$$\| q\,Q(D)v\| \leqq C \| P(D)v \|, \qquad v \in S, \tag{3.32}$$

holds as well. From this and (3.9) it follows that $D(P_0) \subseteq D(q\,Q_0)$.

To prove (3.32), let v be any function in S and set $f = P(D)v$. Then

$$qQ(D)v = q[F^{-1}(Q/P) * f]. \tag{3.33}$$

Now by (3.11) $D^{\mu}(Q/P)$ is in L^1 whenever $a|\mu| + c > n$. By hypothesis this holds for any $|\mu| \geqq k_0$. Thus the hypotheses of Theorem 3.1 are satisfied for $Tf = q\,Q(D)v$. The result follows from inequality (3.19).

Proof of Theorem 2.13. For $R > 0$ set

$$\begin{aligned} q_R(x) &= q(x), \qquad |x| \leqq R \\ &= 0, \qquad |x| > R. \end{aligned}$$

Let ψ be a function in C_0^∞ satisfying $0 \leqq \psi \leqq 1$, $\psi(x) = 1$ for $|x| < 1$, $\psi(x) = 0$ for $|x| > 2$. Set $\psi_r(\xi) = \psi(\xi/r)$, $r > 0$. Now by (3.33)

$$\begin{aligned} Tf = qQ(D)v &= q_R[F^{-1}(\psi_r Q/P) * f] + q_R\{F^{-1}[(1-\psi_r)Q/P] * f \\ &+ (q - q_R)[F^{-1}(Q/P) * f] = T_1 f + T_2 f + T_3 f. \end{aligned}$$

Now for each R and r, T_1 is a compact operator on L^p. For by Lemma 3.2 $A\,f = F^{-1}(\psi_r\,Q/P) * f$ is a compact operator from L^p to $C(\bar{\Omega})$, where Ω is the sphere $|x| < R$. Since q is locally in L^p, q_R is a bounded operator from $C(\bar{\Omega})$ to L^p. Hence T_1 is compact on L^p.

Next I claim that T_2 and T_3 are bounded operators on L^p and their bounds can be made as small as desired by taking R and r sufficiently large. For by Theorem 3.1

$$\| T_2 \| \leqq C K_3 [M_{\alpha,p}(q_R)]^{1/p},$$

where K_3 is a bound for derivatives of $(1 - \psi_r)Q/P$ of order $n + 1$. But by (3.11) these derivatives are as small as we like by taking r sufficiently large.

Since

$$M_{\alpha,p}(q_R) \leqq M_{\alpha,p}$$

the same is true of $\| T_2 \|$. Next I claim that there is a γ satisfying

$$\alpha < \gamma < p(n - k_0) - n$$

and

$$(3.34) \qquad \int_{|x-y|<1} |q(x)|^p |x-y|^\gamma dx \to 0 \text{ as } |y| \to \infty.$$

This means that

$$(3.35) \qquad M_{\gamma,p}(q - q_R) \to 0 \text{ as } R \to \infty.$$

Now by Theorem 3.1

$$\| T_3 \| \leqq C(K_1 + K_2)[M_{\gamma,p}(q - q_R)]^{1/p},$$

which shows that $\| T_3 \|$ can be made as small as desired by taking R sufficiently large. Thus T is the limit in norm of compact operators on L^p. Hence T is compact. This implies that $q\,Q_0$ is P_0-compact.

Thus to complete the proof we merely must prove (3.34). It is obvious for $p(n - k_0) > n$. Otherwise we have by Hölder's inequality

$$(3.36) \qquad \int_{|x-y|<1} |q(x)|^p |x-y|^\gamma dx\, dx \leqq \left(\int_{|x-y|<1} |q(x)|^p dx \right)^{1/s}$$

$$\left(\int_{|x-y|<1} |q(x)|^p |x-y|^{\gamma s'} dx \right)^{1/s'}$$

Take s so large that $\alpha + (|\alpha|/s) < p(n - k_0) - n$. Then set $\gamma = \alpha + (|\alpha|/s)$. This gives $\alpha = \gamma s'$. Hence

$$(3.37) \qquad \int_{|x-y|<1} |q(x)|^p |x-y|^\gamma dx \leqq \int_{|x-y|<1} |q(x)|^p dx^{\,1/s} \, [M_{\alpha,p}(q)]^{1/s'}.$$

Thus (3.34) follows from (2.7). This completes the proof.

Proof of Theorem 2.14. By Theorem 3.1.

$$(3.38) \qquad \Sigma \| a_j(x) Q_j(D) v \| \leqq C(\| P(D)v \| + \| v \|), \qquad v \in S,$$

from which we conclude $D(P_0) \subseteq D(a_j(x)Q_{j0})$, and consequently $D(P_0) \subseteq D(L_0)$. Moreover, on $D(P_0)$

$$L_0 - P_0 = \Sigma [a_j(x) - a_j] Q_{j0},$$

and each operator $[a_j(x) - a_j]Q_{j0}$ is P_0 compact by Theorem 2.13. Thus the same is true for $L_0 - P_0$ and the proof is complete.

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